Exam A (Part II) **Name**

Important: Show all your work in the spaces provided. Present your solutions in a well organized, neat, and legible way. Calculators may be used *only* in elementary computational and trig mode, and not in calculus mode (even for exploratory purposes).

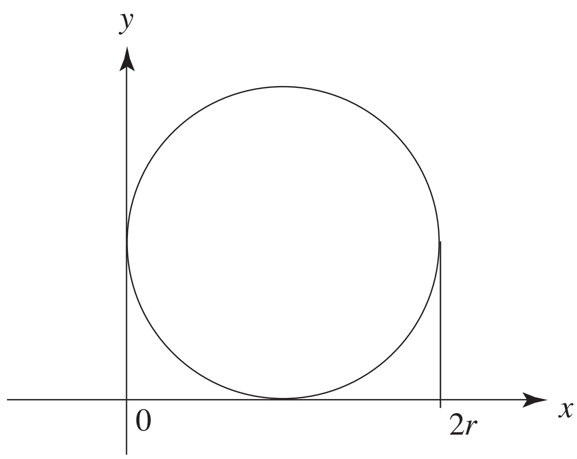
1. Let $y = f(x) = x^3 + 3x^2 - 24x$. Find the critical numbers for this function, determine the intervals over which its graph is increasing or decreasing, find the values of x that give rise to (local) maximum and minimum values.

2. The sum $(4)^{\frac{1}{2}} \cdot \frac{1}{1000} + \left(4 + \frac{1}{1000}\right)^{\frac{1}{2}} \frac{1}{1000} + \left(4 + \frac{2}{1000}\right)^{\frac{1}{2}} \frac{1}{1000} + \cdots + \left(8 + \frac{999}{1000}\right)^{\frac{1}{2}} \frac{1}{1000}$ follows the pattern that the first three terms and the last term establish. Study the sum involved in the definition of $\int_a^b f(x)\,dx$. For what n, $a = x_0, x_1, \ldots, x_{n-1}, x_n = b$, and $y = f(x)$ is the sum above of this form? Find a number C that closely approximates the value of the sum above.

$C =$

3. You are given a function $y = f(x)$ that is continuous on the interval $[a, b]$. True or false: Is there always a function $y = F(x)$ such that $\int_a^b f(x)\,dx = F(b) - F(a)$? If this is false, say why it is false. If it is true, explain how such a function $y = F(x)$ can be defined for the given $y = f(x)$.

4. A circle of radius r is positioned as shown in the figure below.



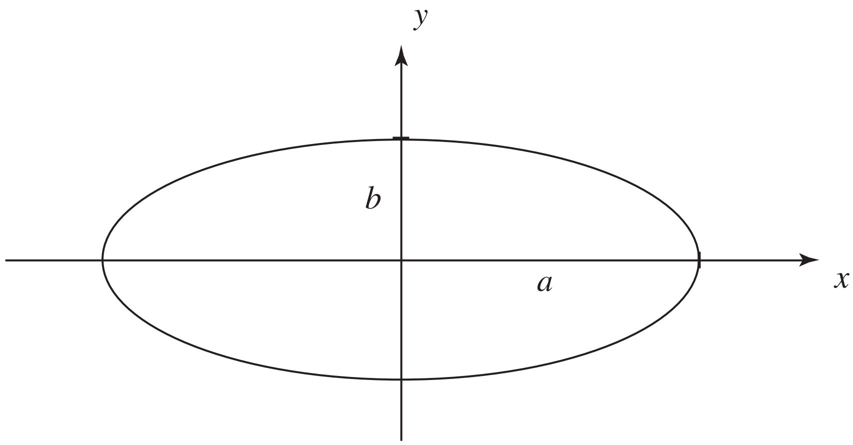
4a. Find the area A under the lower half of the circle and above the interval $[0, 2r]$ by using the geometry of the situation.

$A =$

4b. Express the area A as a definite integral.

$A = \int$

5. The upper half of the ellipse of Figure 3 is the graph of the function $f(x) = \frac{b}{a}\sqrt{a^2 - x^2}$. Revolve the entire right side of the ellipse of the figure one complete revolution around the y-axis.



Express the resulting volume V as a definite integral in two different ways. (One way involves the disc method and the variable y, and the other way the shell method and the variable x.)

$V = \int$

$V = \int$

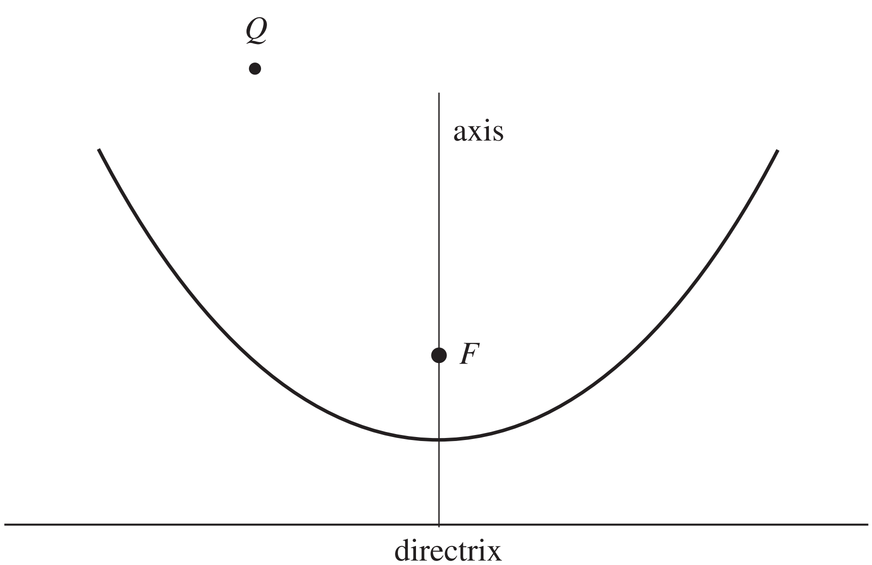
Find the volume that has been described by evaluating one of the two integrals.

V =

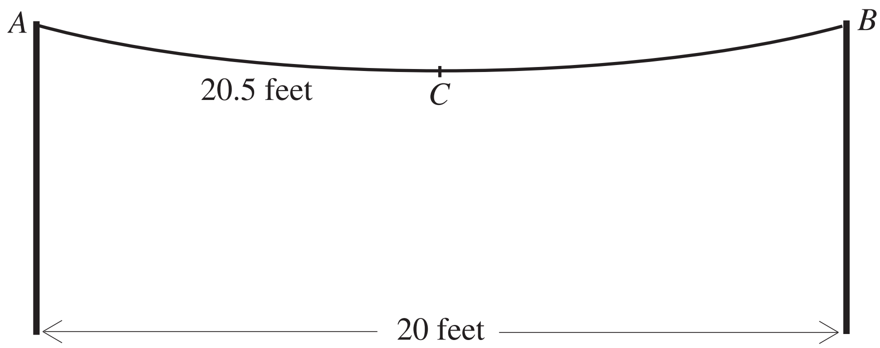
6. One half the circumference of a circle of radius r is πr. Use this fact to evaluate the definite integral $\int_{-r}^{r} \frac{1}{\sqrt{r^2 - x^2}}\, dx$.

$$\int_{-r}^{r} \frac{1}{\sqrt{r^2 - x^2}}\, dx =$$

7. The figure below depicts a parabolic mirror. A beam of light goes through the point Q, strikes the mirror and is reflected through the focus F. Explain why the path of approach of the beam towards the mirror is parallel to the focal axis of the parabola. [Hint: By Fermat's principle, the path the beam from Q to the mirror to F must be the path of least time. Since the medium is the same throughout, the path needs to be the shortest possible path.]



8. A clothesline 20.5 feet long is attached to two poles at points A and B. The poles are 20 feet apart. The straight line that connects A and B is horizontal. See Figure 6. (The clothesline is completely flexible and inelastic. Compared to the loads that it supports, it is of negligible weight.)



A. A laundry bag with 120 wet socks (60 pairs) weighing a total of 26 pounds has been attached to the clothesline at its midpoint C.

i. Show that in the situation just described, the point C is 2.25 feet below the line segment AB.

ii. Let T be the tension in the line and draw a diagram of the forces acting at C. Combine the force diagram and the geometry to show that $T \approx 59.22$ pounds

B. The 120 socks are taken out of the laundry bag and suspended individually on the clothesline at approximately 6 socks per horizontal foot. Since 120 socks weigh 26 pounds, 6 socks weigh 1.3 pounds. Since the distance between the posts is 20 feet, the clothesline supports $w = 1.3$ pounds per foot. Let $d = 10$ be one-half the distance between the posts. The midpoint C of the clothesline has shifted from its previous situation. It is now s feet below the segment AB. Place a coordinate system into the figure so that the origin is at C and the x-axis is parallel to the segment AB.

i. Let L be the length of the part of the clothesline from C to B. With the coordinate system as specified, we know that the curve of the cable is the graph of the function $f(x) = \frac{s}{d^2}x^2$. Explain why the length L of the cable from C to B is given by

$$L = \int_0^d \sqrt{1 + \left(\tfrac{2s}{d^2}x\right)^2}\, dx \text{ (nothing cosmic here, just mention a basic fact).}$$

ii. Recall the approximation

$$\int_0^d \sqrt{1 + \left(\tfrac{2s}{d^2}x\right)^2}\, dx \approx d + \tfrac{2}{3}\left(\tfrac{s}{d}\right)^2 d - \tfrac{2}{5}\left(\tfrac{s}{d}\right)^4 d.$$

Use it and the quadratic formula to derive an estimate for s^2. From this, deduce the approximation $s \approx 1.96$ feet.

iii. Use the conclusion of (ii) to show that the maximum tension $T_{\max}$ and the minimum tension $T_{\min}$ in the clothesline are 35.62 pounds and 33.16 pounds respectively.

Formulas and expressions:

$$\int u\,dv = uv - \int v\,du \qquad x = r\cos\theta \quad y = r\sin\theta \qquad Ay'' + By' + Cy = 0$$

$$y = D_1 e^{r_1 x} + D_2 e^{r_2 x} \qquad y = D_1 e^{2x} + D_2 x e^{2x} \qquad y = e^{ax}(D_1 \cos bx + D_2 \sin bx)$$

$$a = \tfrac{d}{1-\varepsilon^2} \qquad b = \tfrac{d}{\sqrt{1-\varepsilon^2}} \qquad a = \tfrac{d}{\varepsilon^2-1} \qquad b = \tfrac{d}{\sqrt{\varepsilon^2-1}} \qquad f'(\theta) = f(\theta)\cdot\tan(\gamma - \tfrac{\pi}{2})$$

$$f(x) = \tfrac{s}{d^2}x^2, \quad \tan\alpha = \tfrac{2s}{d}, \quad T(x) = w\sqrt{\left(\tfrac{d^2}{2s}\right)^2 + x^2}, \quad T_d = wd\sqrt{\left(\tfrac{d}{2s}\right)^2 + 1}, \quad T_0 = \tfrac{wd^2}{2s}.$$

$$ab\pi \qquad a^2 = b^2 + c^2 \qquad \tfrac{x^2}{a^2} + \tfrac{y^2}{b^2} = 1 \quad \kappa = \tfrac{ab\pi}{T} \qquad F = ma \quad f(t)\cdot r = I\cdot\alpha(t) \qquad \tfrac{d}{dx}\sin x = \cos x$$

$$\int_a^b \sqrt{1 + f'(x)^2}\,dx \qquad \int_a^b \pi f(x)^2\,dx \quad 2\pi\int_a^b f(x)\sqrt{1 + f'(x)^2}\,dx$$